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The center of a system of non-parallel forces

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Abstract

From a discrete system $\mathscr{F}$ of applied forces given by a collection of vectors $\mathbf{F}_k$ applied to corresponding points P_k, a new system $\mathbf{Q}\mathscr{F}$ can be obtained through a rotation by $\mathbf{Q}$ of all $\mathbf{F}_k$ without changing P_k. In this note we examine invariant properties of $\mathscr{F}$ under arbitrary rotations. We also examine invariant properties of the family $\mathbf{Q}\mathscr{F}$ when all rotations share a fixed axis, giving a coordinate-free approach to the results of Kolosov (1927).

1. Introduction

Consider a system of N forces each one parallel to the direction $\mathbf{g}$ and with intensity f_k, $k = 1, \ldots, N$. Thus, $\mathbf{F}_k = f_k\mathbf{g}$ and if we assume $\Sigma_{k=1}^{N} f_k \neq 0$, we know that

$$\mathbf{r} = \frac{\sum_{k=1}^{N} f_k \mathbf{r}_k}{\sum_{k=1}^{N} f_k} \tag{1}$$

delivers a point C (on the central axis of the system), given by its position vector $\mathbf{r}$ relative to an origin O in the Euclidean space $\mathscr{E}$. Here, as usual, we are considering force as an applied vector, consisting of a pair $(\mathbf{F}_k, P_k)$, with $\mathbf{r}_k$ the position of P_k relative to O.

We recall that $\mathbf{r}$ depends on $\mathbf{F}_k$ as well as on $\mathbf{r}_k$, and in Mechanics sometimes what matters is the subjacent structure of a set of sliding vectors. In this case, the central axis concept plays an important role. Recall that the central axis is the (straight) line defined by points P for which $\boldsymbol{M}_P = \lambda\mathbf{R}$, i.e., the total moment $\mathbf{M}_P$ with respect to P is parallel to the resultant $\mathbf{R}$ ($\mathbf{R} \neq \mathbf{0}$, $\mathbf{R} = \Sigma_{k=1}^{N} \mathbf{F}_k$).

The point C, the center of a system of parallel forces, satisfies:

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(a) for $\boldsymbol{\rho}_k$ the position of P_k relative to C, $\Sigma_{k=1}^{N} f_k \boldsymbol{\rho}_k = \mathbf{0}$; C is the unique point with this property;
(b) C is invariant under rotations: it does not change if we consider a new system $(\mathbf{QF}_k, P_k)$, where $\mathbf{Q}$ is a fixed proper orthogonal tensor. Thus, the central axes of $(\mathbf{F}_k, P_k)$ and $(\mathbf{QF}_k, P_k)$ meet at C.

In the dynamic of rigid bodies, under a constant gravitational force field, the fact that C coincides with the center of mass simplifies the analysis of the motion. Relatively to a rigid body the gravitational force is a 'live load', as it is seen to rotate as the body rotates in space.

We consider the translation space $\mathscr{V}$ of $\mathscr{E}$ endowed with a vector product $\times$. We define the tensor product $\mathbf{a} \otimes \mathbf{b}$ of two vectors by the rule $(\mathbf{a} \otimes \mathbf{b})\mathbf{u} = (\mathbf{b} \cdot \mathbf{u})\mathbf{a}$ of its action on an arbitrary vector $\mathbf{u}$, where $\mathbf{b} \cdot \mathbf{u}$ is the scalar product of $\mathbf{b}$ and $\mathbf{u}$. From definition it follows that

$$(\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a})\mathbf{u} = -(\mathbf{a} \times \mathbf{b}) \times \mathbf{u},$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u} \in \mathscr{V}$.

The vector space of all linear transformations of $\mathscr{V}$ into $\mathscr{V}$ is denoted by Lin. We call tensor any element of Lin. Thus $\mathbf{a} \otimes \mathbf{b}$ is a tensor. The transpose of $\mathbf{A} \in \text{Lin}$ is denoted $\mathbf{A}^{\mathrm{T}}$ and $(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a}$.

The astatic load of the system relative to a point P is (Truesdell and Noll, 1992, for example)

$$\mathbf{A}_P = \sum_{k=1}^{N} \mathbf{r}_k \otimes \mathbf{F}_k,$$

with $P + \mathbf{r}_k = P_k$, and we see that the moment of the system relative to P equals minus twice the vector corresponding to the skew part of $\mathbf{A}_P$ (for any skew tensor $\mathbf{W}$ its vector $\mathbf{w}$ is defined by $\mathbf{Wu} = \mathbf{w} \times \mathbf{u}$ holding for all $\mathbf{u} \in \mathscr{V}$). Thus, if $\mathbf{M}_P$ is the system moment relative to P, we have, for any $\mathbf{u} \in \mathscr{V}$, $(\mathbf{A}_P^{\mathrm{T}} - \mathbf{A}_P)\mathbf{u} = \mathbf{M}_P \times \mathbf{u}$.

If we compute the astatic load of a parallel system relative to its center we have

$$\mathbf{A}_C = \sum_{k=1}^{N} \boldsymbol{\rho}_k \otimes f_k \mathbf{g} = \left(\sum_{k=1}^{N} f_k \boldsymbol{\rho}_k \right) \otimes \mathbf{g} = \mathbf{0}.$$

In this note we are going to analyse (arbitrary) systems of forces

$$\mathscr{F} = \{(\mathbf{F}_k, P_k), k = 1, \ldots, N\}$$

and their corresponding rotated systems

$$\mathbf{Q}\mathscr{F} = \{(\mathbf{QF}_k, P_k), k = 1, \ldots, N\},$$

with $\mathbf{Q}$ a rotation (orthogonal and proper). We will investigate the invariant properties of the family $\mathbf{Q}\mathscr{F}$ in general, i.e., those properties that are shared by all families $\mathbf{Q}\mathscr{F}$, and in particular those invariant properties under the restricted condition of all rotations having the same axis.

We start our analysis considering systems $\mathscr{F}$ with vanishing astatic load relative to a point. In this case $\mathbf{Q}\mathscr{F}$ share all nice geometric properties of a system of parallel forces as stated in our first theorem.

In Sections 4 and 5 we recall the generalized notion of center C for an arbitrary system $\mathscr{F}$ and we define another remarkable point which we call the astatic center S. We show that S is invariant under rotations, we establish necessary and sufficient conditions for the general and restricted invariance of C, obtaining along this process an estimate for the distance between C and S. Finally we reconsider the work of Kolosov (1927) under our coordinate free approach.

2. Properties of the astatic load

Consider a system of forces $\mathscr{F}$ and its astatic load $\mathbf{A}_P = \Sigma_{k=1}^{N} \mathbf{r}_k \otimes \mathbf{F}_k$ relative to P. As the moment of $\mathscr{F}$ relative to P, $\mathbf{M}_P = \Sigma_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k$, is equal to minus twice the vector corresponding to the skew part of $\mathbf{A}_P$, the astatic load embodies more information on the geometry of the system than its moment. Because the moment does not change for the subjacent structure of a set of sliding vectors, we expect that the astatic load will play an important role in our analysis. The following lemma is easily proved.

Lemma (transport of the astatic load). If $\mathbf{u}$ is the position of Q relative to P, i.e., $Q = P + \mathbf{u}$, then

$$\mathbf{A}_P = \mathbf{A}_Q + \mathbf{u} \otimes \mathbf{R}, \tag{2}$$

where $\mathbf{R} = \Sigma_{k=1}^{N} \mathbf{F}_k$.

This formula shows that $\mathbf{A}_P = \mathbf{A}_Q$ in systems with null resultant. Moreover, in contrast with the case of the transport of the moment, if $\mathbf{A}_P = \mathbf{A}_Q$ for P and Q distinct, then $\mathbf{R} = \mathbf{0}$ follows necessarily. We can consider the astatic load as a tensor field defined over all of $\mathscr{E}$. Thus the vanishing of the resultant $\mathbf{R}$ implies that the astatic load is spatially *constant* (and conversely).

From now on we admit that $\mathscr{F}$ always has non-zero resultant $\mathbf{R}$.

Returning to a system of parallel forces $\mathscr{F}$, we observe that the center C of the system also has the following two properties:

(a) It is a point on the central axis for which the trace of $\mathbf{A}_C$ is zero:
(b) It is a point C for which the astatic load has minimum norm.

We will show that if we consider (a) or (b) as *defining* a point in $\mathscr{E}$ for a general system of forces $\mathscr{F}$, the definition makes sense and delivers each one a single point for $\mathscr{F}$, usually two *distinct* points. We name them C (the center of $\mathscr{F}$) and S (the astatic center of $\mathscr{F}$), respectively.

Given $\mathscr{F}$, consider the rotated system $\mathbf{Q}\mathscr{F}$. Let $\mathbf{A}_P$ be the astatic load of $\mathscr{F}$ with respect to P. From the identity $\mathbf{a} \otimes \mathbf{Qb} = (\mathbf{a} \otimes \mathbf{b})\mathbf{Q}^{\mathrm{T}}$, it follows that the astatic load $\mathbf{A}(\mathbf{Q})_P$ of $\mathbf{Q}\mathscr{F}$ with respect to P is

$$\mathbf{A}(\mathbf{Q})_P = \mathbf{A}_P \mathbf{Q}^{\mathrm{T}}. \tag{3}$$

This shows how the astatic load changes with the rotation, in their places, of the forces of $\mathscr{F}$.

3. Systems with vanishing astatic load

For a parallel system $\mathscr{F}$, the central axes of all $\mathbf{Q}\mathscr{F}$ intersect at C. Does it hold for an arbitrary $\mathscr{F}$? To see it, assume that $\mathbf{R} = \Sigma_{k=1}^{N} \mathbf{F}_k \neq \mathbf{0}$ and that the central axes of all rotated systems intersect at a point P. Then the skew part of $\mathbf{A}_P \mathbf{Q}^{\mathrm{T}}$ has its vector parallel to $\mathbf{QR}$, i.e.,

$$(\mathbf{A}_P \mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}_P^{\mathrm{T}})\mathbf{QR} = \mathbf{0}. \tag{4}$$

This is equivalent to $(\mathbf{QA}_P^{\mathrm{T}}\mathbf{Q})\mathbf{R} = \mathbf{A}_P\mathbf{R}$ for all rotations $\mathbf{Q}$. But this holds if and only if $\mathbf{A}_P = \mathbf{0}$, as it is shown below.

Lemma 1. Let $\mathbf{R} \neq \mathbf{0}$ be a vector and $\mathbf{A}$ a tensor, i.e., $\mathbf{A} \in \mathrm{Lin}$. Then $(\mathbf{QA}^{\mathrm{T}}\mathbf{Q})\mathbf{R} = \mathbf{AR}$ for all rotations $\mathbf{Q}$ if and only if $\mathbf{A} = \mathbf{0}$.

Proof. As $\mathbf{A} = \mathbf{0}$ implies trivially that (4) holds, we must prove the necessity. For $\mathbf{Q} = \mathbf{I}$, where $\mathbf{I}$ is the identity, we have $\mathbf{A}^{\mathrm{T}}\mathbf{R} = \mathbf{AR}$. Choose $\mathbf{u} \in \mathscr{V}$; then $\mathbf{QA}^{\mathrm{T}}\mathbf{QR} \cdot \mathbf{u} = \mathbf{A}^{\mathrm{T}}\mathbf{R} \cdot \mathbf{u}$. In particular, if $\mathbf{u}$ is not parallel to $\mathbf{R}$ and if $\mathbf{Q}$ has axis parallel to $\mathbf{u}$, i.e., $\mathbf{Qu} = \mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{u}$, we have $\mathbf{A}^{\mathrm{T}}(\mathbf{QR}-\mathbf{R}) \cdot \mathbf{u} = 0$. But as $\mathbf{QR}-\mathbf{R}$, for all $\mathbf{Q}$'s fixing $\mathbf{u}$, generates the subspace orthogonal to $\mathbf{u}$, it follows that $\mathbf{u}$ is an eigenvector for $\mathbf{A}$. Thus, $\mathbf{A}$ is a multiple of the identity and now (4) clearly implies $\mathbf{A} = \mathbf{0}$.

As a corollary of this lemma, it follows that if $\mathbf{AQ}^{\mathrm{T}}$ is symmetric for all rotations $\mathbf{Q}$, then (4) holds for any choice of $\mathbf{R}$ and $\mathbf{A} = \mathbf{0}$ follows as a consequence. We explicitly state:

Corollary 1. If $\mathbf{A} \in \mathrm{Lin}$ and if $\mathbf{AQ}$ is symmetric for all rotations $\mathbf{Q}$, then $\mathbf{A} = \mathbf{0}$.

Hence if for a system $\mathscr{F}$ we define a point in $\mathscr{E}$ as the common intersection of the central axes of families $\mathbf{Q}\mathscr{F}$, this definition makes sense only for those particular systems admitting a zero astatic load. These systems can be neatly characterized through the astatic load relative to an arbitrary point P. As $\mathbf{A}_P + \mathbf{u} \otimes \mathbf{R} = \mathbf{0}$ for a suitable non-zero vector $\mathbf{u}$, $\mathbf{A}_P$ has rank one and its kernel is orthogonal to $\mathbf{R}$.

For the following result, we recall that for any system $\mathscr{F}$ the inner product of its resultant $\mathbf{R}$ with the moment $\mathbf{M}_P$ of $\mathscr{F}$ relative to any point P is constant: $\mathbf{R} \cdot \mathbf{M}_P$ is the scalar invariant of $\mathscr{F}$, in this particular case *invariant* referring to a constant in space.

Theorem 1. Let $\mathscr{F}$ be a system for which $\mathbf{R} \neq \mathbf{0}$. Then the following are equivalent:

(i) The central axes of all $\mathbf{Q}\mathscr{F}$ intersect at a common point.
(ii) Relative to a point P, the moments of all $\mathbf{Q}\mathscr{F}$ vanish.
(iii) The scalar invariant of all $\mathbf{Q}\mathscr{F}$ is zero.
(iv) The astatic load of $\mathscr{F}$ equals zero at one point.

Proof. We know that (i) $\Leftrightarrow$ (iv). As (ii) means that $\mathbf{A}_P\mathbf{Q}^{\mathrm{T}}$ is symmetric for all rotations, implying $\mathbf{A}_P = \mathbf{0}$ by Corollary 1, (ii) and (iv) are equivalent too.

We recall that any tensor $\mathbf{F} \in \mathrm{Lin}$ admits a decomposition $\mathbf{F} = \mathbf{VR}$ as the product of $\mathbf{V}$ symmetric times a rotation $\mathbf{R}$ (see Martins and Podio-Guidugli, 1980). Also we recall that such decomposition is unique if we assume $\det \mathbf{F} > 0$ and $\mathbf{V}$ positive, as in the classical use of the polar decomposition theorem in continuum mechanics.

For (iii) we start observing that for any point P, $\mathbf{A}_P\mathbf{Q}^{\mathrm{T}}$ is symmetric for a suitable rotation; thus the whole family of systems $\mathbf{Q}\mathscr{F}$ can have constant scalar invariant $\mathbf{M} \cdot \mathbf{R}$, if it is zero.

To say that $\mathbf{Q}\mathscr{F}$ has zero scalar invariant means that the axis of $\mathbf{A}_P\mathbf{Q}^{\mathrm{T}} - \mathbf{QA}_P^{\mathrm{T}}$ is orthogonal to $\mathbf{QR}$. If we choose an orthogonal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{R}\}$ for $\mathscr{V}$, then

$$(\mathbf{A}_P\mathbf{Q}^{\mathrm{T}} - \mathbf{QA}_P^{\mathrm{T}})\mathbf{Qe} = \alpha\mathbf{QR}, \tag{5}$$

holds for all rotations (where α depends on $\mathbf{Q}$ and $\mathbf{e}$), because the moment vector lies in the plane generated by $\{\mathbf{Qe}, \mathbf{Qf}\}$. In (5) the dot product with $\mathbf{Qf}$ gives

$$\mathbf{A}_P\mathbf{e} \cdot \mathbf{Qf} = \mathbf{A}_P^{\mathrm{T}}\mathbf{Qe} \cdot \mathbf{f}. \tag{6}$$

For all rotations $\mathbf{Q}'$ with axis $\mathbf{e}$, (6) reduces to $\mathbf{A}_P\mathbf{e}\cdot\mathbf{Q}'\mathbf{f} = \mathbf{A}_P^{\mathrm{T}}\mathbf{e}\cdot\mathbf{f}$. The right-hand-side being constant, this implies that $\mathbf{A}_P\mathbf{e}$ is normal to the plane generated by all $\mathbf{Q}'\mathbf{f}$. Thus $\mathbf{A}_P\mathbf{e}$ is parallel to $\mathbf{e}$ for any vector orthogonal to $\mathbf{R}$. For a rotation $\mathbf{Q}''$ of angle $\pi/2$ along $\mathbf{R}$ for which $\mathbf{Q}''\mathbf{e} = \mathbf{f}$ and $\mathbf{Q}''\mathbf{f} = -\mathbf{e}$, one gets from (6) $-\mathbf{A}_P\mathbf{e}\cdot\mathbf{e} = \mathbf{A}_P^{\mathrm{T}}\mathbf{f}\cdot\mathbf{f}$, showing that $\mathbf{A}_P\mathbf{e} = \mathbf{0}$. Using this information we can rewrite (5) as

$$-\mathbf{A}_P^{\mathrm{T}}\mathbf{Q}\mathbf{e} = \alpha\mathbf{R}, \tag{7}$$

holding for an arbitrary rotation. This means that the range of $\mathbf{A}_P^{\mathrm{T}}$ is contained in the span of $\{\mathbf{R}\}$, implying $\mathbf{A}_P^{\mathrm{T}} = \mathbf{R}\otimes\mathbf{u}$ for some vector $\mathbf{u}$. Hence the astatic load of any such system vanish for some point in $\mathscr{E}$. Thus, (iii) and (iv) are equivalent.

Another geometrically nice property for a system $\mathscr{F}$ with moment $\mathbf{M}_P$ relative to a point P is to suppose that $\mathbf{Q}\mathbf{M}_P$ is the corresponding moment (relative to P) for the system $\mathbf{Q}\mathscr{F}$, for any rotation $\mathbf{Q}$; this implies that $(\mathbf{A}_P\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}_P^{\mathrm{T}})\mathbf{Q}\mathbf{M}_P = \mathbf{0}$ holds for all rotations $\mathbf{Q}$. By Lemma 1, if $\mathbf{M}_P \neq \mathbf{0}$ it follows that $\mathbf{A}_P = \mathbf{0}$, contradicting $\mathbf{M}_P \neq \mathbf{0}$. But if $\mathbf{M}_P = \mathbf{0}$, $\mathbf{Q}\mathbf{M}_P = \mathbf{0}$ and now we have by hypothesis $\mathbf{A}_P\mathbf{Q}^{\mathrm{T}}$ symmetric for all rotations. Thus $\mathbf{A}_P = \mathbf{0}$. Observe that it makes sense to state this property even if $\mathbf{R} = \mathbf{0}$.

Recall that $\mathscr{F}$ is said to be an *equilibrated system* if both $\mathbf{R} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$ ($\mathbf{R} = \mathbf{0}$ implies that the moment $\mathbf{M}$ is the same for all points). Thus, $\mathbf{Q}\mathscr{F}$ is also equilibrated for all rotations $\mathbf{Q}$ if and only if $\mathbf{R} = \mathbf{0}$ and the astatic load is zero for one point (hence for all points).

Given P and Q distinct points in $\mathscr{E}$, let $Q = P+\mathbf{u}$. The system defined by $\mathscr{S} = \{(-f\mathbf{u}, P), (f\mathbf{u}, Q)\}$ has resultant $\mathbf{R} = \mathbf{0}$ and its (constant) astatic load is given by $\mathbf{u}\otimes f\mathbf{u} = f\mathbf{u}\otimes\mathbf{u}$. Thus $\mathscr{S}$ is an equilibrated system, and from the clear additive property for the union of two systems of forces, for any $\mathbf{E}\in\mathrm{Lin}$, $\mathbf{E}$ symmetric, we can associate an equilibrated system $\mathscr{F}$ with corresponding astatic load equal to $\mathbf{E}$.

On the other hand, if we choose $\mathbf{v}$ a vector orthogonal to $\mathbf{u}$, $\mathbf{v}\neq\mathbf{0}$, the system $\mathscr{C} = \{(-f\mathbf{v}, P), (f\mathbf{v}, Q)\}$ has astatic load $\mathbf{A} = \mathbf{u}\otimes f\mathbf{v} = f\mathbf{u}\otimes\mathbf{v}$. Systems as $\mathscr{C}$ ($f\neq 0$) are usually called a couple. Another couple is given by $\mathscr{C}' = \{(f\mathbf{u}, P), (-f\mathbf{u}, N)\}$ with $N = P+\mathbf{v}$. Its astatic load is $-\mathbf{v}\otimes f\mathbf{u} = -f\mathbf{v}\otimes\mathbf{u}$, and for the system $\mathscr{C}\bigcup\mathscr{C}'$, its astatic load is $f(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u})$, a constant skew field. Now clearly to any $\mathbf{W}\in\mathrm{Lin}$, $\mathbf{W}$ skew symmetric, we can associate a system $\mathscr{F}$ with null resultant and astatic load equal to $\mathbf{W}$. Finally we observe that if $\mathscr{F}$ is an equilibrated system, we can join to $\mathscr{F}$ three equilibrated systems of the form $\mathscr{S} = \{(-\mathbf{f}\mathbf{u}, P), (f\mathbf{u}, P+\mathbf{u})\}$ in order to obtain a new equilibrated system with corresponding zero astatic load.

4. The astatic center S

The astatic center of $\mathscr{F}$ ($\mathbf{R}\neq\mathbf{0}$) is the unique point S at which $\mathbf{A}_S$ has minimum Euclidean norm. Choose P in $\mathscr{E}$. As $\mathbf{A}_P - \mathbf{s}\otimes\mathbf{R}$ is the astatic load for $Q = P+\mathbf{s}$ and because the set of tensors $\mathbf{u}\otimes\mathbf{R}$ with $\mathbf{u}$ varying on $\mathscr{V}$ is a subspace of Lin, the problem of minimizing the norm of the astatic load has a unique solution given by the condition: find $\mathbf{s}\in\mathscr{V}$ such that $(\mathbf{A}_P - \mathbf{s}\otimes\mathbf{R})\cdot(\mathbf{u}\otimes\mathbf{R}) = 0$ holds for all $\mathbf{u}$. Thus from

$$\mathrm{tr}\,[(\mathbf{A}_P - \mathbf{s}\otimes\mathbf{R})(\mathbf{R}\otimes\mathbf{u})] = (\mathbf{A}_P\mathbf{R} - R^2\mathbf{s})\cdot\mathbf{u} = 0$$

we get

$$\mathbf{s} = \frac{\mathbf{A}_P\mathbf{R}}{R^2}, \tag{8}$$

for the position of S with respect to P. In particular, if $\mathbf{A}_S$ is the astatic load of $\mathscr{F}$ with respect to its astatic center, (8) shows that $\mathbf{A}_S\mathbf{R} = \mathbf{0}$. Moreover, as for $\mathbf{Q}\mathscr{F}$, $\mathbf{A}(\mathbf{Q})_S = \mathbf{A}_S\mathbf{Q}^{\mathrm{T}}$ and the corresponding resultant is $\mathbf{QR}$, $\mathbf{A}_S\mathbf{R} = \mathbf{A}_S\mathbf{Q}^{\mathrm{T}}\mathbf{QR} = \mathbf{0}$ shows that the astatic center is invariant under rotations. In general the astatic center does not belong to the central axis of the system.

5. The center of an arbitrary system of forces

The center C of a system $\mathscr{F}$ ($\mathbf{R} \neq \mathbf{0}$) is defined as the unique point C on its central axis at which $\mathbf{A}_C$ is traceless. Assume $(\mathbf{A}_C - \mathbf{A}_C^{\mathrm{T}})\mathbf{R} = \mathbf{0}$ and $\operatorname{tr}\mathbf{A}_C = 0$. Referring to any point P with $C = P + \mathbf{r}$ we have

$$\left.\begin{aligned} (\mathbf{A}_P - \mathbf{r}\otimes\mathbf{R} - \mathbf{A}_P^{\mathrm{T}} + \mathbf{R}\otimes\mathbf{r})\mathbf{R} &= \mathbf{0} \\ \operatorname{tr}\mathbf{A}_P &= \mathbf{r}\cdot\mathbf{R} \end{aligned}\right\}. \tag{9}$$

The moment condition gives $(\mathbf{A}_P - \mathbf{A}_P^{\mathrm{T}})\mathbf{R} - R^2\mathbf{r} + (\mathbf{r}\cdot\mathbf{R})\mathbf{R} = \mathbf{0}$. Thus

$$\mathbf{r} = \frac{1}{R^2}[(\mathbf{A}_P - \mathbf{A}_P^{\mathrm{T}})\mathbf{R} + (\operatorname{tr}\mathbf{A}_P)\mathbf{R}], \tag{10}$$

gives the position of C relative to P as a function of $\mathbf{A}_P$.

As we know that two force systems are said to be *statically equivalent* if they have the same resultant force $\mathbf{R}$ and produce the same total moment relative to a point $P \in \mathscr{E}$ (hence equal to total moment about any point of $\mathscr{E}$). When $\mathbf{R} \neq \mathbf{0}$, any system $\mathscr{F}$ is statically equivalent to a 'force-wrench', a system comprising a force $\mathbf{R}$ acting on any point of the central axis of $\mathscr{F}$ and a couple $\mathscr{C}$ with its moment equal to the total moment $\mathbf{M}_P$ of $\mathscr{F}$ with respect to any point P on the central axis of $\mathscr{F}$: $\mathbf{M}_P // \mathbf{R}$.

From (10) the point $C' = P + (1/R^2)(\mathbf{A}_P - \mathbf{A}_P^{\mathrm{T}})\mathbf{R}$ belongs to the central axis of $\mathscr{F}$. Moreover, as $(\mathbf{A}_P - \mathbf{A}_P^{\mathrm{T}})\mathbf{R}$ is orthogonal to $\mathbf{R}$, C' is the point of the central axis at minimum distance from P.

We know that the astatic center is invariant under rotations. As $\mathbf{A}_S\mathbf{R} = \mathbf{0}$, the distance δ from S to the central axis of $\mathscr{F}$ is

$$\delta = \frac{1}{R^2}\|\mathbf{A}_S^{\mathrm{T}}\mathbf{R}\|,$$

which is *not* invariant under rotations. However the symmetric and non-negative tensor $\mathbf{A}_S\mathbf{A}_S^{\mathrm{T}}$ is clearly invariant.

Let Λ be the largest eigenvalue of $\mathbf{A}_S\mathbf{A}_S^{\mathrm{T}}$. Then the distance $\delta(\mathbf{Q})$ from S to the central axis of $\mathbf{Q}\mathscr{F}$ can be estimated by

$$\delta(\mathbf{Q}) \leqslant \frac{\sqrt{\Lambda}}{R},$$

and the central axis of all rotated systems $\mathbf{Q}\mathscr{F}$ cross a ball with center at S and radius $(1/R)\sqrt{\Lambda}$.

For rotations $\mathbf{Q}$ for which $\mathbf{A}_S\mathbf{Q}^T$ is symmetric, $\delta(\mathbf{Q}) = 0$ and $\mathbf{M}(\mathbf{Q})_S = \mathbf{0}$. The inner product $\mathbf{M}(\mathbf{Q})_S \cdot (\mathbf{Q}\mathbf{R}/R)$ depends on the rotation $\mathbf{Q}$, and to analyse this dependence we assume $\mathbf{A}_S = \mathbf{V}$ symmetric, without loss of generality. We call $\mathbf{g}$ the unit vector of $\mathbf{R}$, i.e., $\mathbf{R} = R\mathbf{g}$. Because $\mathbf{V}\mathbf{R} = \mathbf{0}$, let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis of $\mathscr{V}$ delivering the spectral decomposition of $\mathbf{V}$. We assume this basis positively oriented and we write $\mathbf{V} = \lambda\mathbf{e} \otimes \mathbf{e} + \beta\mathbf{f} \otimes \mathbf{f}$.

Each skew tensor $\mathbf{W}$ is related to $\mathbf{w} \in \mathscr{V}$ through the condition $\mathbf{w} \times \mathbf{u} = \mathbf{W}\mathbf{u}$ holding for all $\mathbf{u} \in \mathscr{V}$. If $\mathbf{W}_i$ is related to $\mathbf{w}_i$ then $2\mathbf{w}_1 \cdot \mathbf{w}_2 = \mathbf{W}_1 \cdot \mathbf{W}_2$ holds. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = \mathbf{Q}\mathbf{u} \times \mathbf{Q}\mathbf{v}$ and $\mathbf{Q}\mathbf{w} \times \mathbf{u} = \mathbf{Q}\mathbf{w} \times \mathbf{Q}\mathbf{Q}^T\mathbf{u} = \mathbf{Q}(\mathbf{w} \times \mathbf{Q}^T\mathbf{u}) = \mathbf{Q}\mathbf{W}\mathbf{Q}^T\mathbf{u}$ holding for all rotations $\mathbf{Q}$ shows that if $\mathbf{w}$ is related to $\mathbf{W}$ then $\mathbf{Q}\mathbf{w}$ is related to $\mathbf{Q}\mathbf{W}\mathbf{Q}^T$.

Let $\mathbf{G}$ be the skew tensor related to $\mathbf{g}$. Then

$$2\mathbf{M}(\mathbf{Q})_S \cdot \mathbf{Q}\mathbf{g} = (\mathbf{A}(\mathbf{Q})_S^T - \mathbf{A}(\mathbf{Q})_S) \cdot \mathbf{Q}\mathbf{G}\mathbf{Q}^T$$

and as $\mathbf{G} = \mathbf{f} \otimes \mathbf{e} - \mathbf{e} \otimes \mathbf{f}$ and $\mathbf{A}(\mathbf{Q})_S = \mathbf{V}\mathbf{Q}^T$, a simple computation shows that

$$2\mathbf{M}(\mathbf{Q})_S \cdot \mathbf{Q}\mathbf{g} = -2\lambda(\mathbf{e} \cdot \mathbf{Q}\mathbf{f}) + 2\beta(\mathbf{f} \cdot \mathbf{Q}\mathbf{e}).$$

By choosing an appropriate $\mathbf{Q}$ we conclude that

$$-|\lambda| - |\beta| \leqslant \mathbf{M}(\mathbf{Q})_S \cdot \mathbf{Q}\mathbf{g} \leqslant |\lambda| + |\beta|,$$

both extreme values attainable for rotations $\mathbf{Q}$ preserving the direction (not the sense) of $\mathbf{g}$, and for which $\mathbf{Q}\mathscr{F}$ is statically equivalent to a wrench through the astatic center S.

The mechanical significance of this result is clear. Let a rigid body B be subjected to a system $\mathscr{F}$ of dead loads (i.e., the vector $\mathbf{F}_k$ is constant) always applied to points P_k fixed in B. Suppose the astatic load relative to the center S of $\mathscr{F}$, fixed relatively to B, small with respect to the product of R with a characteristic distance.Then to admit the load system equivalent to its resultant applied to S can be justified.

Finally let us see the simplifications gained in the case of a planar system of forces. As all P_k and all $\mathbf{F}_k$ are lying in a plane, the system is statically equivalent to a single force acting in this plane. For the center C, $\mathbf{A}_C$ is symmetric and has the direction normal to the plane in its kernel. Thus the spectral representation of $\mathbf{A}_C$ is $\mathbf{A}_C = \lambda\mathbf{e} \otimes \mathbf{e} - \lambda\mathbf{f} \otimes \mathbf{f}$, for $\{\mathbf{e}, \mathbf{f}\}$ an orthonormal basis for the plane. From C, the position of the astatic center S is given by $R^2\mathbf{s} = \mathbf{A}_C\mathbf{R}$, showing that C and S coincide if and only if $\mathbf{A}_C = \mathbf{0}$. We recall that if $\mathbf{R}_f$ is a reflection in the plane along the axis $\mathbf{e}$, i.e., $\mathbf{R}_f\mathbf{e} = \mathbf{e}$, then for any rotation $\mathbf{Q}$, $\mathbf{R}_f\mathbf{Q}$ is a reflection and $\mathbf{R}_f\mathbf{Q} = \mathbf{Q}^T\mathbf{R}_f$ holds, as a simple computation shows it. Thus, if $\mathbf{Q}$ corresponds to a rotation of angle 2θ, and if $\mathbf{S}$ is a rotation of angle θ, $\mathbf{R}_f\mathbf{Q} = \mathbf{R}_f\mathbf{S}\mathbf{S} = \mathbf{S}^T\mathbf{R}_f\mathbf{S}$ is a reflection of axis $\mathbf{S}^T\mathbf{e}$.

The expression of $\mathbf{A}_C$ just obtained, namely $\mathbf{A}_C = \lambda\mathbf{e} \otimes \mathbf{e} - \lambda\mathbf{f} \otimes \mathbf{f}$ shows that $\mathbf{A}_C$ is a multiple of a *planar reflection*. Moreover $\mathbf{A}_C\mathbf{Q}^T$ is again a multiple of a *planar reflection*, whenever $\mathbf{Q}$ has axis orthogonal to the plane. Thus all central axes of the restricted family $\mathbf{Q}\mathscr{F}$ meet at C. This result will be generalized in Section 7 (Corollary 2).

6. Restricted invariance of the center of forces

In this section we will investigate the invariance of the center C of a family $\mathbf{Q}\mathscr{F}$ under the restriction that all $\mathbf{Q}$ have the same axis. Thus, let us suppose that the center C of $\mathscr{F}$ is invariant under rotations about a unit direction $\mathbf{g}$. The assumed invariance implies

$$(\mathbf{A}_C\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}_C^{\mathrm{T}})\mathbf{QR} = \mathbf{0}, \qquad (11)$$
$$\mathrm{tr}(\mathbf{A}_C\mathbf{Q}^{\mathrm{T}}) = 0.$$

As $\mathbf{Q}$ has the representation $\mathbf{Q} = \mathbf{I} + (\sin\theta)\mathbf{G} + (1-\cos\theta)\mathbf{G}^2$, where the vector of the skew tensor $\mathbf{G}$ is $\mathbf{g}$, we conclude that $0 = \mathrm{tr}(\mathbf{A}_C\mathbf{Q}^{\mathrm{T}}) = \mathrm{tr}\,\mathbf{A}_C + (\sin\theta)\mathbf{A}_C\cdot\mathbf{G} + (1-\cos\theta)\mathbf{A}_C\cdot\mathbf{G}^2$. This implies $\mathbf{A}_C\cdot\mathbf{G} = 0$ and $\mathbf{A}_C\cdot\mathbf{G}^2 = 0$. Recall that if $\mathbf{h}$ is the vector of the skew tensor $\mathbf{H}$, then $2\mathbf{g}\cdot\mathbf{h} = \mathbf{G}\cdot\mathbf{H}$. In particular if $\mathbf{M}_C$ is the moment of $\mathscr{F}$ with respect to C, as $\mathbf{M}_C$ is the vector of $\mathbf{A}_C^{\mathrm{T}} - \mathbf{A}_C$, then $2\mathbf{M}_C\cdot\mathbf{g} = (\mathbf{A}_C^{\mathrm{T}} - \mathbf{A}_C)\cdot\mathbf{G} = -2\mathbf{A}_C\cdot\mathbf{G}$. Thus, $\mathbf{A}_C\cdot\mathbf{G} = 0$ is equivalent to $\mathbf{M}_C\cdot\mathbf{g} = 0$. As $\mathbf{G}^2 = \mathbf{g}\otimes\mathbf{g} - \mathbf{I}$, $\mathbf{A}_C\cdot\mathbf{G}^2 = 0$ is equivalent to $\mathbf{A}_C\mathbf{g}\cdot\mathbf{g} = 0$. Hence $\mathbf{M}_C\cdot\mathbf{g} = 0$ and $\mathbf{A}_C\mathbf{g}\cdot\mathbf{g} = 0$ are necessary conditions for the assumed invariance. As these were obtained from $(11)_2$ we do not expect that they are sufficient. In fact, if we change the second condition for the stronger condition $\mathbf{A}_C\mathbf{g} = \mathbf{0}$, then $\mathbf{M}_C\cdot\mathbf{g} = 0$ and $\mathbf{A}_C\mathbf{g} = \mathbf{0}$ are shown sufficient for the restricted invariance as we start to prove.

Suppose $\mathbf{M}_C = \mathbf{0}$, implying that $\mathbf{A}_C$ is symmetric. From $(11)_1$ $\mathbf{A}_C\mathbf{R} = \mathbf{Q}\mathbf{A}_C\mathbf{QR}$ implies $\mathbf{A}_C\mathbf{R}\cdot\mathbf{g} = \mathbf{A}_C\mathbf{QR}\cdot\mathbf{g}$, or $(\mathbf{R} - \mathbf{QR})\cdot\mathbf{A}_C\mathbf{g} = 0$. If $\mathbf{R}$ is not parallel to $\mathbf{g}$ we can conclude that $\mathbf{A}_C\mathbf{g}$ is parallel to $\mathbf{g}$, thus, equal to zero because $\mathbf{A}_C\mathbf{g}\cdot\mathbf{g} = 0$; but if $\mathbf{R}$ and $\mathbf{g}$ are parallel, from $(11)_1$ $\mathbf{A}_C\mathbf{R} = \mathbf{Q}\mathbf{A}_C\mathbf{R}$ shows that $\mathbf{A}_C\mathbf{g}$ is parallel to $\mathbf{g}$. Hence $\mathbf{A}_C\mathbf{g} = \mathbf{0}$ in both cases and $\mathbf{A}_C$ acts on the normal plane to $\mathbf{g}$ as a multiple of a plane reflection. In fact, because $\mathbf{g}$ belongs to the kernel of $\mathbf{A}_C$ and $\mathbf{A}_C$ is traceless, it can be written as $\mathbf{A}_C = \lambda\mathbf{e}\otimes\mathbf{e} - \lambda\mathbf{f}\otimes\mathbf{f}$ for some λ and for its eigenbasis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. On the other hand, $\mathbf{A}_C = \lambda\mathbf{e}\otimes\mathbf{e} - \lambda\mathbf{f}\otimes\mathbf{f}$ satisfies both eqns (11), corresponding to $\mathbf{M} = \mathbf{0}$ for all $\mathbf{Q}\mathscr{F}$.

Suppose now $\mathbf{M} \neq \mathbf{0}$. From $\mathbf{M}\cdot\mathbf{g} = 0$ and because $\mathbf{M}$ and $\mathbf{R}$ ($\neq\mathbf{0}$) are parallel, we must have $\mathbf{R}\cdot\mathbf{g} = 0$ as well. From $(11)_1$ $\mathbf{A}_C\mathbf{R} = \mathbf{Q}\mathbf{A}_C^{\mathrm{T}}\mathbf{QR}$, and as $\mathbf{A}_C\mathbf{R} = \mathbf{A}_C^{\mathrm{T}}\mathbf{R}$ we have $\mathbf{A}_C^{\mathrm{T}}\mathbf{R}\cdot\mathbf{g} = \mathbf{A}_C^{\mathrm{T}}\mathbf{QR}\cdot\mathbf{g}$, or $\mathbf{A}_C\mathbf{g}\cdot(\mathbf{R} - \mathbf{QR}) = 0$. Hence $\mathbf{A}_C\mathbf{g}$ is parallel to $\mathbf{g}$ and the necessary condition $\mathbf{A}_C\mathbf{g}\cdot\mathbf{g} = 0$ gives $\mathbf{A}_C\mathbf{g} = \mathbf{0}$.

Thus, whenever (11) holds for all rotations, we have $\mathbf{A}_C\mathbf{g} = \mathbf{0}$ and $\mathbf{A}_C\cdot\mathbf{G} = \mathbf{M}_C\cdot\mathbf{g} = 0$ as necessary conditions for the invariance of C for the restricted family $\mathbf{Q}\mathscr{F}$. We already know that they are sufficient if $\mathbf{M}_C = \mathbf{0}$. Now we show that they are sufficient also if $\mathbf{M}_C \neq \mathbf{0}$. As $\mathbf{M}_C$ and $\mathbf{R}$ are parallel by hypothesis, $\mathbf{R}\cdot\mathbf{g} = 0$; as $\mathbf{R}$ is not zero if we choose an orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$, with $\mathbf{e}$ parallel to $\mathbf{R}$, the skew part of $\mathbf{A}_C$ is a multiple of $\mathbf{g}\otimes\mathbf{f} - \mathbf{f}\otimes\mathbf{g}$. Thus, $\mathbf{A}_C = \beta(\mathbf{g}\otimes\mathbf{f} - \mathbf{f}\otimes\mathbf{g}) + \mathbf{E}$, where $\mathbf{E}$ is the tensor $\mathbf{E} = \frac{1}{2}(\mathbf{A}_C + \mathbf{A}_C^{\mathrm{T}})$. From $\mathbf{0} = \mathbf{A}_C\mathbf{g} = -\beta\mathbf{f} + \mathbf{Eg}$ we have $\mathbf{E} = \mathbf{E}' + \beta\mathbf{f}\otimes\mathbf{g} + \beta\mathbf{g}\otimes\mathbf{f}$ for some symmetric and traceless $\mathbf{E}'$ for which $\mathbf{E}'\mathbf{g} = \mathbf{0}$ ($\mathbf{E}'$ is a multiple of a plane reflection on the $\mathbf{e}$–$\mathbf{f}$-plane). Thus $\mathbf{A}_C = \mathbf{E}' + 2\beta\mathbf{g}\otimes\mathbf{f}$ and because both $\mathbf{E}'$ and $2\beta\mathbf{g}\otimes\mathbf{f}$ satisfies (11) the condition is sufficient and we have:

Theorem 2. For a system $\mathscr{F}$ let $\mathbf{g}$ be a unit vector, $\mathbf{G}$ the skew tensor associated with $\mathbf{g}$, and let $\mathbf{A}_C$ be the astatic load relative to its center. In addition to its defining properties, namely $\mathrm{tr}\,\mathbf{A}_C = 0$ and $(\mathbf{A}_C - \mathbf{A}_C^{\mathrm{T}})\mathbf{R} = \mathbf{0}$ (or the moment $\mathbf{M}_C$ relative to C parallel to $\mathbf{R}$), then $\mathbf{A}_C\mathbf{g} = \mathbf{0}$ and $\mathbf{A}_C\cdot\mathbf{G} = 0$ are the necessary and sufficient conditions for the invariance of the center of the family $\mathbf{Q}\mathscr{F}$, where all $\mathbf{Q}$ have $\mathbf{g}$ as axis.

Observe that if $\mathbf{A}_C$ is not symmetric, $\mathbf{R}\cdot\mathbf{g} = 0$ is necessary for the restricted invariance.

The proof that $\mathbf{A}_C\mathbf{g} = \mathbf{0}$ and $\mathbf{M}_C\cdot\mathbf{g} = 0$ are sufficient for the restricted invariance relies on expressing $\mathbf{A}_C$ on a special basis of $\mathscr{V}$. Let us now see how this condition is translated if we represent $\mathbf{A}_C$ in an orthonormal basis $\{\mathbf{e}_k\}$ with $\mathbf{e}_3$ being the common axis of all rotations. If $\mathbf{A}_C$ is symmetric, we know that $\mathbf{M} = \mathbf{0}$ for all $\mathbf{Q}\mathscr{F}$. But $\mathbf{A}_C\mathbf{e}_3 = \mathbf{0}$ and $\operatorname{tr}\mathbf{A}_C = 0$ imply that

$$[\mathbf{A}_C] = \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is necessarily the representation of $\mathbf{A}_C$. In this case the components (X, Y, Z) of $\mathbf{R} = X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3$ can be arbitrary.

In the case when $\mathbf{A}_C$ is not symmetric, $Z = 0$ is a necessary condition for the restricted invariance and $\mathbf{M}_C$ is parallel to $\mathbf{R}$. Then we know that $\mathbf{A}_C$ can be represented as $\mathbf{A}_C = \mathbf{E}' + 2\beta\mathbf{g}\otimes\mathbf{f}$, with $\mathbf{g} = \mathbf{e}_3$ and $\mathbf{f}$ orthogonal to both $\mathbf{g}$ and the direction of $\mathbf{R}$. Moreover, $\mathbf{E}'$ is symmetric, traceless and satisfies $\mathbf{E}'\mathbf{e}_3 = \mathbf{0}$. Thus, as $(Y, -X, 0)$ is parallel to $\mathbf{f}$, $\mathbf{A}_C = \mathbf{E}' + d(\mathbf{e}_3 \otimes (Y\mathbf{e}_1 - X\mathbf{e}_2))$ for a constant d and the corresponding matrix representation of $\mathbf{A}_C$ is

$$[\mathbf{A}_C] = \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ dY & -dX & 0 \end{bmatrix}.$$

Both representations can be recorded as above with the understanding that $d = 0$ if and only if $\mathbf{M}_C = \mathbf{0}$. As $\mathbf{A}_C$ is the astatic load relative to the system center, $\mathbf{M}_C = dX\mathbf{e}_1 + dY\mathbf{e}_2$ reminds us that when $d \neq 0$, then necessarily $Z = 0$.

Finally, as

$$[\mathbf{Q}] = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a rotation with axis $\mathbf{e}_3$ ($\cos\theta = c, \sin\theta = s$), a simple computation shows that the last row of $\mathbf{A}_C\mathbf{Q}^{\mathrm{T}}$ is $(d(Xs + Yc), -d(Xc - Ys), 0)$, while $\mathbf{QR}$ equals $(Xc - Ys, Xs + Yc)$, showing that the modulus of $\mathbf{M}(\mathbf{Q})_C$ does not change.

7. The approach of Kolosov

We now comment on the work of Kolosov (1927) under a new perspective. In it, necessary and sufficient conditions for the invariance of C under rotations about a fixed direction are given in terms of four equations relating the components of the resultant $\mathbf{R}$, and the components of the astatic load with respect to the origin of a Cartesian frame where the z-axis coincides with the axis of the rotations. They are:

$$\begin{aligned} XA_{zz} &= ZA_{zx}, \\ YA_{zz} &= ZA_{zy}, \\ Z(A_{xx} + A_{yy}) &= XA_{xz} + YA_{yz}, \\ Z(A_{xy} - A_{yx}) &= YA_{xz} - XA_{yz}. \end{aligned} \tag{12}$$

He establishes these formulae by use of complex numbers to express the x- and y-coordinates of C, the z-coordinate of C, and the scalar invariant of $\mathscr{F}$ in terms of the forces of $\mathscr{F}$. Now it is easy to treat the action of $\mathbf{Q}$ as a multiplication by $e^{i\theta}$ and the four conditions express the invariance of C.

But if the system $\mathscr{F}$ is given, $\mathbf{A}_P$, (10) and the transport of the astatic load give $\mathbf{A}_C$. Hence, from $\mathbf{A}_C\mathbf{g} = \mathbf{0}$ and $\mathbf{A}_C \cdot \mathbf{G} = 0$ we get four necessary and sufficient conditions for the restricted invariance. Kolosov formulae give no hint for the structure of $\mathbf{A}_C$, but it turns out that $\mathbf{A}_C\mathbf{g} = \mathbf{0}$ and $\mathbf{A}_C \cdot \mathbf{G} = 0$ expressed in coordinates using the formula, give a rather intricate system of four non-linear relations for the coordinates of $\mathbf{A}_P$ and $\mathbf{R}$. We can explore the canonical form for $\mathbf{A}_C$ in order to get (12) explicitly by considering the cases $Z = 0$ and $Z \neq 0$ separately. As a matter of fact, when $\mathbf{Z} = 0$ we have $A_{xz} = A_{yz} = A_{zz} = 0$ as the only conditions imposed by (12).

Let us start supposing $Z = 0$. From our previous analysis we are looking for $\mathbf{r}$ with components x, y, z such that the matrix of $\mathbf{A}_C$ given by

$$\begin{bmatrix} A_{xx}-xX & A_{xy}-xY & A_{xz} \\ A_{yx}-yX & A_{yy}-yY & A_{yz} \\ A_{zx}-zX & A_{zy}-zY & A_{zz} \end{bmatrix}$$

corresponds to the astatic load relative to the invariant center of $\mathscr{F}$. Thus the last column has to be zero. We can impose $\operatorname{tr}\mathbf{A}_C = 0$ and the symmetry of the 2×2 submatrix defined on the upper corner of $\mathbf{A}_C$ by solving a 2×2 system (we need $X^2 + Y^2 \neq 0$, which holds by hypothesis). It is also easy to see that z can be chosen such that $(A_{zx} - zx)X = -(A_{zy} - zY)Y$ to have the corresponding moment parallel to $\mathbf{R}$. Thus, if $Z = 0$, it is enough to verify that $\mathbf{A}_P\mathbf{g} = \mathbf{0}$.

Now suppose $Z \neq 0$. As before we look for $\mathbf{r}$ such that $\mathbf{A}_C$ has the corresponding canonical form. We have now $\mathbf{A}_C$ given by

$$\begin{bmatrix} A_{xx}-xX & A_{xy}-xY & A_{xz}-xZ \\ A_{yx}-yX & A_{yy}-yY & A_{yz}-yZ \\ A_{zx}-zX & A_{zy}-zY & A_{zz}-zZ \end{bmatrix}.$$

Choose now x, y and z to meet the requirement of null column, $x = (A_{xz}/Z)$, $y = (A_{yz}/Z)$, and $z = (A_{zz}/Z)$. From $A_{zx} - (A_{zz}/Z)X = 0$ and $A_{zy} - (A_{zz}/Z)Y = 0$ we reproduce $(12)_{1,2}$. From the symmetry $A_{yx} - (A_{yz}/Z)X = A_{xy} - (A_{xz}/Z)Y$ we get $(12)_4$ and the third equation comes from the zero trace condition: $A_{xx} - (A_{xz}/Z)X + A_{yy} - (A_{yz}/Z)Y = 0$.

Finally we state explicitly a consequence of Theorem 2, whose proof is now easy after the later development.

Corollary 2. If all forces in the system $\mathscr{F}$ with resultant $\mathbf{R} \neq \mathbf{0}$ are *parallel* to a fixed plane, then the system center C of $\mathscr{F}$ and of all $\mathbf{Q}\mathscr{F}$, for restricted rotations $\mathbf{Q}$ along axis normal to the plane, coincide.

Proof. As $\mathbf{A}_P = \Sigma_{k=1}^{N} \mathbf{r}_k \otimes \mathbf{F}_k$ and as $\mathbf{A}_P\mathbf{g} = \mathbf{0}$ for $\mathbf{g}$ orthogonal to the plane, the result follows.

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References

Kolosov, G.V., 1927. Sur le centre des forces non parallèles. C.R. Acad. Sci. Paris 184 (14-26), 1010–1014.

Martins, L.C., Podio-Guidugli, P., 1980. An elementary proof of the polar decomposition theorem. American Mathematical Monthly 87 (4), 288–290.

Truesdell, C.A., Noll, W., 1992. The Nonlinear Field Theories of Mechanics, 2nd ed. Springer-Verlag, Berlin, New York.